Classical Ising model in various dimensions.

In $d=1$, the Ising model orders at $T=0$ while it is disordered at any $T>0$. We showed this explicitly in 210A using the transfer matrix method, where we also discussed the heuristic domain-wall argument that shows that domain walls proliferate at any non-zero T. Let's recall the argument: the free energy contribution from a single domain wall $\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow$ scales as

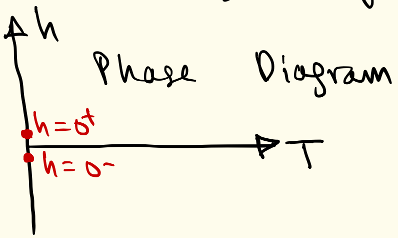
$$\Delta F \sim \underbrace{2J}_{\Delta E} - \underbrace{T\log(L)}_{T\Delta S} \Rightarrow \text{entropy dominates energy at}$$

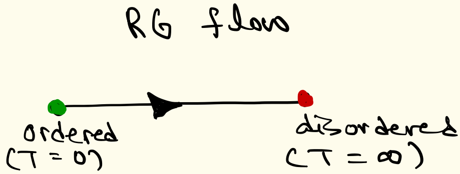
any $T>0$. One can consider an improved version this argument that allows us to calculate the density of domain walls at temperature T. Assuming that there are x number of domain walls, lets calculate free energy as f^n of x and then optimize over x.

$$\Delta F \sim 2Jx - T\log\left[{}^{L}C_{x}\right] \quad \text{where} \quad {}^{N}C_{m} = \frac{N!}{m!(N-m)!}$$

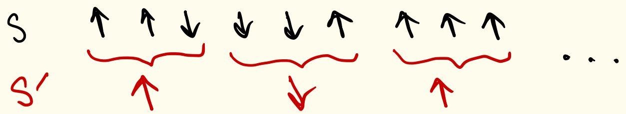
Minimizing w.r.t. x yields $\frac{x}{L} \sim e^{-2J/T}$, as expected. The distance between domain walls $\sim \frac{1}{x} \sim e^{2J/T}$ is the correlation length.

All of this implies that temperature T is a relevant variable of the $T=0$ ordered fixed point. The phase diagram and the RG flow are thus given by:





Let's do an explicit RG calculation to re-derive these results. As discussed earlier, the basic idea of RG is to define coarse-grained variables in terms of the original variables. The most natural procedure would be the 'majority rule':



To implement this, we define a coarse-grained Hamiltonian H' for s' as:

$$e^{-H'(s')} = \mathrm{Tr}_s \prod_{\text{blocks}} T(s' \mid s \in \text{block})\, e^{-H(s)}$$

where $T(s'; s \in \text{block}) = 1$ if s' has same sign as that of $\sum_i s_i$.

$= 0$ otherwise.

Note that due to $\sum_{s'} T(s'; s \in \text{block}) = 1$, the partition f^n is unchanged after coarse-graining.

$$Z' = \mathrm{Tr}_{s'} \, \mathrm{Tr}_s \prod_{\text{blocks}} T(s' | s \in \text{block}) \, e^{-H(s)}$$

$$= \mathrm{Tr}_s \, e^{-H(s)} = Z.$$

However, this coarse-graining is a bit unwieldy to implement analytically. Although, it's fairly straightforward to implement numerically, e.g., in Monte Carlo.

Lets consider a different RG scheme where we trace out every other spin to obtain an effective Hamiltonian for the left-over spins:

$$Z = \mathrm{Tr}\left[e^{\beta \sum_i S_{2i}(S_{2i-1} + S_{2i+1})} \right]$$

Tracing out all even numbered spins

$$e^{-H'} = \mathrm{Tr}_{S \in \text{even sites}} \; e^{\beta \sum_i S_{2i}(S_{2i-1}+S_{2i+1})}$$

$$= \prod_i 2\cosh\left[\beta (S_{2i-1} + S_{2i+1})\right]$$

We make two important observations at this point:

(i) The effective Hamiltonian H' of the untraced spins has the same symmetries as the original Hamiltonian H. In particular, it has the $\mathbb{Z}_2$ sym $S \to -S$.

(ii) The effective Hamiltonian H' is short-ranged (i.e. local), similar to H.

The first property is guranteed in general by the fact that partial tracing preserves all on-site ("internal") symmetries. In principle, it can break point-group/translation syms. if the tracing procedure does not respect those.

The second property is a consequence of the fact that we are tracing out short-wavelength d.o.f. It's not gauranteed in general, but it's essential for the RG to make sense.

Due to sym. $s \to -s$ of H', we should be able to express

$$2\cosh[\beta(s_{2i-1}+s_{2i+1})] = A\, e^{\beta' s_{2i-1} s_{2i+1}}$$

To solve for A and more importantly, β', we can get a set of two eqns. by substituting $s_{2i-1} = s_{2i+1}$ and $s_{2i-1} = -s_{2i+1}$:

$$2\cosh(2\beta) = A\, e^{\beta'} \qquad (s_{2i+1} = s_{2i-1})$$

$$2 = A\, e^{-\beta'} \qquad (s_{2i+1} = -s_{2i-1})$$

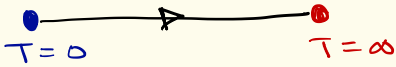
$$\Rightarrow \quad e^{2\beta'} = \cosh(2\beta)$$

Since $\cosh(2\beta) = \dfrac{e^{2\beta}+e^{-2\beta}}{2} < e^{2\beta}$ (for $\beta > 0$)

$$\Rightarrow \quad \beta' < \beta \Rightarrow \text{effective temperature increases under RG monotonically.}$$

There are two fixed points: $\beta = 0$ and $\beta = \infty$.

⇒ The RG flow looks like:



as expected, i.e. the $T=0$ ordered phase is unstable to infinitesimal temperature.

One may re-express the RG flow as:

$$v' = v^2 \qquad \text{where} \quad v = \tanh(\beta) \in [0,1]$$

Since we traced out every other spin, the new correlation length is $\frac{1}{2} \times$ original correlation length ⇒

$$\xi(v') = \frac{\xi(v)}{2}$$

$$\Rightarrow \quad \xi(v^2) = \frac{\xi(v)}{2}$$

$$\Rightarrow \quad \xi(v) \sim -\frac{1}{\log(v)} = -\frac{1}{\log(\tanh(\beta))}$$

$$\sim e^{2\beta} \quad \text{as } \beta \to \infty.$$

This is indeed what we found using the exact solution using transfer matrix although a simpler derivation is to use the low-temperature expansion:

$$\langle S_0 \; S_x \rangle = \frac{\mathrm{Tr} \prod_i e^{-\beta S_i S_{i+1}} \; S_0 S_x}{\mathrm{Tr} \prod_i e^{-\beta S_i S_{i+1}}}$$

$$\sim \mathrm{Tr} \prod_i \left[1 + \tanh(\beta) S_i S_{i+1} \right] S_0 S_x$$

In the trace in the numerator, the only non-vanishing contribution comes from the string

$$\mathrm{Trace}\left[S_0 S_x \tanh(\beta) \; S_0 S_1 \; \tanh(\beta) S_1 S_2 \; \cdots \; \tanh(\beta) S_{x-1} S_x \right]$$

$$\Rightarrow \quad \langle S_0 S_x \rangle \sim \left[\tanh(\beta) \right]^x \sim e^{-x/\xi}$$

$$\text{where} \quad \xi \sim \frac{-1}{\log(\tanh(\beta))} \sim e^{2\beta}$$

Differential form of RG flow:

$$v(b) = v^b .$$

Writing $b = e^{dl} \simeq 1 + dl$,

$$\Rightarrow \quad v(1+dl) = v^{1+dl} = v\left[1 + dl \log v \right]$$

$$\Rightarrow \quad \frac{dv}{dl} = v \log v$$

Substituting $v = \tanh(\beta)$, one finds

$$\frac{dT}{dl} = \frac{T^2}{2} \quad \text{i.e. marginally relevant} \qquad (T = 1/\beta)$$

As expected, temperature increases along the RG flow. Integrating the above equation

$$\left.\frac{-1}{T(l)}\right|_0^l = l$$

$$\Rightarrow \quad \frac{-1}{T(l)} + \frac{1}{T} = \frac{l}{2}, \quad \text{where } T(l=0) = T$$

When l is large $T \simeq \infty$. Furthermore,

$\xi(T) = e^l \xi(T(l))$ since at each infinitesimal RG step, ξ decreases by a factor e^{dl}. When l is large $\xi \simeq$ lattice spacing $\ll \xi(T(l=0)) = \xi(T)$

$$\Rightarrow \quad \frac{1}{T} = \frac{l}{2} \sim \log(\xi)$$

$$\Rightarrow \quad \xi \sim e^{2/T}$$

The large correlation length can be thought of as consequence of the marginal relevance of T.